NOTE. The editors desire that contributors send in good problems for solution. Let us have a great variety of problems for solutions in the various departments. Also send in solutions; we prefer to publish solutions prepared by contributors, rather than publish our own. We have neither time, inclination, nor ability to solve every problem proposed in the MONTHLY. However, if every contributor will give a little time to the problems each month, by united effort, there will be few problems remain unsolved. We have recently republished a number of unsolved problems, and we shall be pleased to have solutions of any of them, or any others that remain unsolved. ED, F.

GEOMETRY.

384. Proposed by S. LEFSHETZ, Universityof Nebraska.

Let ABC be a triangle, O a circle tangent to its three sides, T a variable tangent of O, which cuts the sides BC, CA, AB in a, b, c. Oa', Ob', Oc' the perpendiculars in O to Oa, Ob, Oc, cutting, respectively, T in points a', b', c'. Prove that Aa', Bb', Cc' meet in a point t, and find the locus of t when T varies. Purely geometrical proofs wanted.

Solution by R. P. BAKER, Iowa City, Iowa.

I. By elementary geometry.

Lemma I. The second tangent from a', $a'K$ to the circle is parallel to BC.

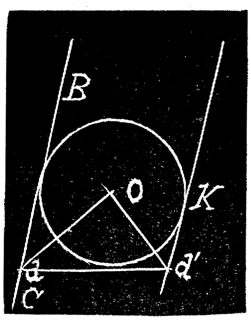

Fig. 1.

For $a'aB$, $aa'A$ are the doubles of the complementary angles $a'aO$ and $aa'O$, and hence supplementary. So $a'K$ is parallel to aB, that is to CB.

Lemma II. If the tangents to a circle from P meet two parallel tangents in Q, Q'; R, R', respectively, and O is the center; then $PQ.PR=OP^2=PQ'.PR'$.

Let M, N be the points of contact of PQ, PR, and S, T of the parallel tangents.

Then $OPQ=OPR$; $2OQP=$supp. $SOM=$supp. $(SOP-MOP)=$supp. $SOP+MOP$.

$2POR=2POT-TON=2POT-(POT-NOP)=POT+MOP=$supp. $SOP+MOP$. Therefore, $OQP=POR$, and the triangles OPQ and RPO are similar.

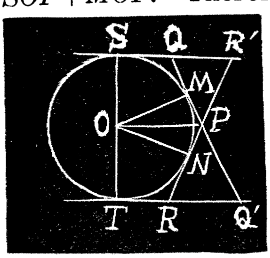

Fig. 2.

Hence $PQ.PR=OP^2$, and by the similar triangles PRQ', $PR'Q$, each is equal to $PQ'.PR'$.

Applying the lemmas to the figure, we have $ab'.aC=ac'.aB$, and the triangles aBb', aCc' are similar, having equal angles at a, and Bb', Cc' are parallel. So for Aa', Bb'.

II. By Brianchon's Theorem.

If the second tangents from b', c' meet at A', we have in AB, BC, CA, $b'A'$, bc, $c'A'$ six tangents to a conic. If the lines are taken in the order written the joins of the cuts of opposite pairs are Bb', Cc', and the line at infinity. Hence Bb', Cc' are parallel, and similarly, the other pairs.

III. Consider the locus of intersections of Bb' and Cc. The pencils at B and C are projective, being in $1:1$ correspondence with the tangents T by a ruler construction. The intersection is at infinity on AB when T coincides with AB, and at infinity on AC when T coincides with AC. It is on BC only if T coincides with BC. From the latter fact the locus is a straight line, and from the two former the line at infinity, the pencils being in perspective.

IV. The generalized problem.

AB, BC, CA, Ta, OI, OJ are six tangents to a conic: Ta cuts AB in c, BC in a, CA in b; OI, OJ have chord of contact IJ: Oa' is the harmonic conjugate of Oa with respect to OI and OJ: a' is its cut with Ta, and so for Ob', Oc'. Then $Aa'Bb'Cc'$ are concurrent on IJ.

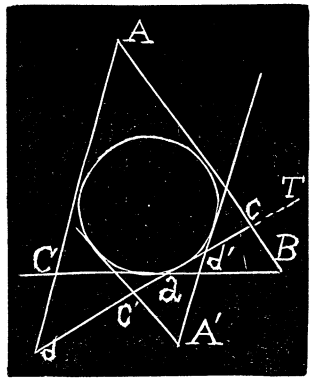

Fig. 3.

Let AB have contact at γ, and OI at κ, IJ at P, OJ at λ. Let PG be the second tangent from P, G its point of contact. $G\gamma$, the polar of P, passes through O, the pole of PIJ. Let it cut IJ at π. Then $PI\pi J$ is harmonic, and so is $P\kappa\gamma\lambda$. The latter is the range determined on the tangent AB by the four tangents PG, OI, AB, OJ. The range on ab being equal, PG cuts this line in c'.

This proves Lemma I in general. Brianchon's Theorem is then applied as in II.

V. By Analytic Geometry.

Take the equation of the circle as

$$x=\frac{1-\lambda^2}{1+\lambda^2};\ y=\frac{2\lambda}{1+\lambda^2}.$$

The tangent at λ is

$$(1-\lambda^2)x+2\lambda y=1+\lambda^2$$

and intersects the tangent at κ in

$$x=\frac{1-\kappa\lambda}{1+\kappa\lambda};\ y=\frac{\kappa+\lambda}{1+\kappa\lambda}.$$

The line through the center perpendicular to the join of this point and the center is:

$$x(1-\kappa\lambda)+y(\kappa+\lambda)=0,$$

and cuts the κ tangent in

$$x=\frac{\kappa+\lambda}{\lambda-\kappa};\ y=\frac{\kappa\lambda-1}{\lambda-\kappa}.$$

The line joining this to the cut of tangents at $(\mu,\ \nu)$ has the slope

$$\frac{\kappa(\lambda+\mu+\nu+\lambda\mu\nu)-(1+\lambda\mu+\mu\nu+\nu\lambda)}{2(\kappa+\lambda\mu\nu)}$$

which is symmetrical in $(\lambda,\ \mu,\ \nu)$. The three lines given by interchanges of $\lambda,\ \mu,\ \nu$ are therefore parallel.

CALCULUS.

312. Proposed by C. N. SCHMALL, New York City.

Given $y^3-3y+x=0$, prove by Maclaurin's theorem, that

$$y=\frac{x}{3}+\frac{x^3}{3^4}+\frac{x^5}{3^6}+\text{etc.}$$

I. Solution by H. PRIME.

Put $u=y^3-3y+x=0$. Then $\partial u/\partial x=1$, $\partial u/\partial y=3y^2-3$; hence, $dy/dx=\frac{1}{3}(1-y^2)=(1+y^2+y^4+y^6+\text{etc.})/3$.

$$d^2y/dx^2=(2y+4y^3+6y^5+\text{etc.})(1+y^2+y^4+\text{etc.})/3^2$$
$$=(2y+6y^3+12y^5+\text{etc.})/3^2.$$

$$d^3y/dx^3=(2+18y^2+60y^4+\text{etc.})(1+y^2+y^4+\text{etc.})/3^3$$
$$=(2+20y^2+80y^4+\text{etc.})/3^3.$$

$$d^4y/dx^4=(40y+320y^3+\text{etc.})(1+y^2+y^4+\text{etc.})/3^4$$
$$=(40y+360y^3+\text{etc.})/3^4.$$

$$d^5y/dx^5=(40+1080y^2+\text{etc.})(1+y^2+y^4+\text{etc.})/3^5$$
$$=(40+1120y+\text{etc.})/3^5;\ \text{etc.}$$

When $x=0$, $(y)=0$. Hence

$(dy/dx)=\frac{1}{3}$, $(d^2y/dx^2)=0$, $(d^3y/dx^3)=2/3^3$, $(d^4y/dx^4)=0$, $(d^5y/dx^5)=40/3^5$, etc. By Maclaurin's formula, $y=x/3+x^3/3^4+x^5/3^6+\text{etc.}$